Vibration of a Three-Layered Ring on Periodic Radial Supports

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The natural frequencies of a three-layered elastic ring on periodic radial supports are obtained by using a wave approach. Two types of support conditions are investigated. When the core is assumed to be viscoelastic, the theory of forced damped normal modes is used to obtain the resonance frequencies and the associated composite modal loss factors of the damped structure. The numerical results show the effects of the thickness ratios and the core shear modulus. The high values of the loss factor in some modes are explained with the aid of the mode shapes of the corresponding elastic ring.

I. Introduction

WITH the increased need for highly damped structures, it has become necessary to augment the damping capacity of a structure with the addition of viscoelastic layers. A constrained damping layer is particularly effective for this purpose. Extensive literature has been devoted for the analysis of beams with unconstrained and constrained layers.

The study of the damping characteristics of a sandwich (constrained layered) ring segment has been presented by Almy and Nelson.[1] They assumed the system to have little damping and used Rayleigh's quotient for computing the resonance frequencies. For an unsupported three-layered ring with a viscoelastic core, Lu et al.[2] have presented theoretical and experimental results in terms of radial driving point impedance. This method has been extended to cover discontinuously constrained damped rings.[3-5] These studies considered a thin walled ring having a finite number of mass segments equally spaced and uniformly attached to its circumference by a thin viscoelastic layer.

DiTaranto[6] has derived the governing differential equations of a three-layered ring using a variational approach. For the damping study, the core was assumed to be viscoelastic with a complex shear modulus and the resonance frequencies and the associated modal loss factors were obtained. A similar analysis for a damped three-layered ring has been carried out by Nelson and Sullivan.[7] The governing equations in this paper were obtained by the equilibrium approach. The condition of inextensibility was used in the equations. Considering the core to be elastic, Sagartz[8,9] and Forrestal and Overmier[10] have analyzed the transient response.

Axially stiffened, layered cylinders are found to constitute the basic structural components in various aerospace applications. A first step toward understanding the dynamic behavior of such structures can be provided by the study of the vibration of layered rings on periodic radial supports. In Ref. 11, the authors have studied the vibration of a two-layered ring on periodic radial supports. The theory of wave propagation was used. The mode shapes of the elastic structure (i.e., both layers elastic) have been presented in Ref. 12.

In this paper, the in-plane vibration of a three-layered ring on periodic radial supports is considered. Differential equations and coupling forces are set up for a periodic element (comprising one bay of the ring) by the variational approach. Treating the core (middle layer) to be elastic, the propagation constant curves and thereby the natural frequencies are obtained. When the core is viscoelastic, the theory of forced damped normal modes[13] is used to obtain the resonance frequencies and the associated composite modal loss factors. The variation of these modal loss factors is explained with the aid of the mode shapes of the corresponding elastic ring. Two types of support conditions are investigated, viz., 1) supports preventing both the radial and tangential displacements of the base layer and 2) supports preventing only the radial displacement. An efficient algorithm[14] based on a two-dimensional Newton-Raphson method is used for computing the resonance frequencies and the modal loss factors of the damped structure. Numerical results are presented for a ring on three supports.

II. Theoretical Analysis

A. Equations of Motion

The following assumptions are made in deriving the differential equations and boundary conditions: 1) the radial displacement remains constant along a particular radial line; 2) the plane cross section of every layer remains plane after deformation; 3) there is no slip at the interfaces of the layers; and 4) the extensional stress in the middle layer is negligible compared to those in the other layers.

Figure 1a shows a three-layered ring on N equispaced radial supports that offer rotational restraint at the midplane of the base layer (layer 3). The rotational stiffness at the supports is represented by k_r.

The total strain energy Π of a single bay (Fig. 1b) of a three-layered ring consists of the individual bending and extensional strain energies of layers 1 and 3, the shear strain energy of layer 2, and the strain energy of the springs at the supports. Using the strain energy expression for a single-layer ring element,[11] Π can be written as

$$\Pi=\frac{K_1}{2}\int_0^\lambda\left(u+\frac{\partial v_1}{\partial\theta}\right)^2\mathrm{d}\theta+\frac{K_3}{2}\int_0^\lambda\left(u+\frac{\partial v_3}{\partial\theta}\right)^2\mathrm{d}\theta$$

$$+\frac{(D_1+D_3)}{2}\int_0^\lambda\left(u+\frac{\partial^2 u}{\partial\theta^2}\right)^2\mathrm{d}\theta+\frac{G2h_2R_2b}{2}\int_0^\lambda\psi^2\mathrm{d}\theta$$

$$+\frac{1}{2}\frac{k_r}{2}\left[\frac{1}{R_3}\left(\frac{\partial u}{\partial\theta}-v_3\right)\right]^2\Bigg|_{\theta=0}$$

$$+\frac{1}{2}\frac{k_r}{2}\left[\frac{1}{R_3}\left(\frac{\partial u}{\partial\theta}-v_3\right)\right]^2\Bigg|_{\theta=\lambda}\qquad(1)$$

where u is the radial displacement, v_j the tangential

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displacement of the midplane, $K_j(=E_jA_j/R_j)$ the extensional stiffness over a length R_j, D_j $(=\frac{2}{3}E_jb\delta_j^3)$ the bending stiffness, E_j Young's modulus, A_j the area of cross section, R_j the radius of the midplane, $\delta_j(=h_j/R_j)$ the nondimensional half thickness, b the width of the ring, h_j the half thickness, $\lambda=(2\pi/N)$ the angle subtended by a single bay at the center of the ring, G the shear modulus of layer 2, and ψ the shear strain in layer 2 with subscript j referring to the jth layer.

In Eq. (1) the quantity $1/R_3(\partial u/\partial\theta - v_3)$ associated with k_r is the local bending slope.

Using assumption 3, ψ can be expressed as[2,6]

$$2h_2\psi = v_1d_1 - v_3d_3 + \frac{\partial u}{\partial\theta}d_2 \tag{2}$$

where $d_1=1-\delta_1$, $d_2=\delta_1+2\,\delta_2+\delta_3$, and $d_3=1+\delta_3$.

The kinetic energy of a single bay T can be written as

$$T=\frac{(m_1+m_2+m_3)}{2}\int_0^\lambda\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}\theta+\frac{m_1}{2}\int_0^\lambda\left(\frac{\partial v_1}{\partial t}\right)^2\mathrm{d}\theta$$

$$+\frac{m_2}{2}\int_0^\lambda\left(\frac{\partial v_2}{\partial t}\right)^2\mathrm{d}\theta+\frac{m_3}{2}\int_0^\lambda\left(\frac{\partial v_3}{\partial t}\right)^2\mathrm{d}\theta \tag{3}$$

where m_j $(=2bh_j\rho_jR_j)$ is the mass of the jth layer over a length R_j with ρ_j as the density of the material of that layer.

The midplane tangential displacement of layer 2, v_2, can be expressed in terms of u, v_1, and v_3 as

$$v_2=\frac{1}{2}\left(v_1d_1+v_3d_3+\frac{\partial u}{\partial\theta}d_4\right) \tag{4}$$

where $d_4=\delta_1-\delta_3$.

By substituting Eqs. (2) and (4) into Eqs. (1) and (3) and setting the variation of Hamiltonian $(T-\Pi)$ equal to zero, one obtains the following differential equations:

$$D\left(u+2\frac{\partial^2u}{\partial\theta^2}+\frac{\partial^4u}{\partial\theta^4}\right)+K_1\left(u+\frac{\partial v_1}{\partial\theta}\right)+K_3\left(u+\frac{\partial v_3}{\partial\theta}\right)$$

$$-gd_2\left(\frac{\partial v_1}{\partial\theta}d_1-\frac{\partial v_3}{\partial\theta}d_3+\frac{\partial^2u}{\partial\theta^2}d_2\right)+m\frac{\partial^2u}{\partial t^2}$$

$$-\frac{m_2d_4}{2}\left(\frac{\partial^3v_1}{\partial\theta\partial t^2}d_1+\frac{\partial^3v_3}{\partial\theta\partial t^2}d_3+\frac{\partial^4u}{\partial\theta^2\partial t^2}d_4\right)=0 \tag{5a}$$

$$K_1\left(\frac{\partial u}{\partial\theta}+\frac{\partial^2v_1}{\partial\theta^2}\right)-gd_1\left(v_1d_1-v_3d_3+\frac{\partial u}{\partial\theta}d_2\right)-m_1\frac{\partial^2v_1}{\partial t^2}$$

$$-\frac{m_2}{2}d_1\left(\frac{\partial^2v_1}{\partial t^2}d_1+\frac{\partial^2v_3}{\partial t^2}d_3+\frac{\partial^3u}{\partial\theta\partial t^2}d_4\right)=0 \tag{5b}$$

$$K_3\left(\frac{\partial u}{\partial\theta}+\frac{\partial^2v_3}{\partial\theta^2}\right)+gd_3\left(v_1d_1-v_3d_3+\frac{\partial u}{\partial\theta}d_2\right)-m_3\frac{\partial^2v_3}{\partial t^2}$$

$$-\frac{m_2}{2}d_3\left(\frac{\partial^2v_1}{\partial t^2}d_1+\frac{\partial^2v_3}{\partial t^2}d_3+\frac{\partial^3u}{\partial\theta\partial t^2}d_4\right)=0 \tag{5c}$$

along with the boundary conditions given by

$$-\left[D\left(u+\frac{\partial^2u}{\partial\theta^2}\right)\right]\delta\left(\frac{\partial u}{\partial\theta}\right)\Big|_0^\lambda$$

$$-\frac{1}{2}\frac{k_r}{R_3^2}\left\{\left[\left(\frac{\partial u}{\partial\theta}-v_3\right)\delta\left(\frac{\partial u}{\partial\theta}\right)\right]\Big|_{\theta=0}\right.$$

$$\left.+\left[\left(\frac{\partial u}{\partial\theta}-v_3\right)\delta\left(\frac{\partial u}{\partial\theta}\right)\right]\Big|_{\theta=\lambda}\right\}=0 \tag{6a}$$

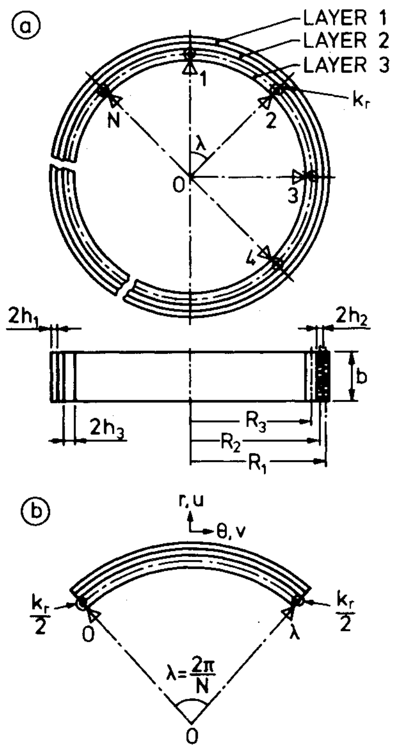

Fig. 1 Three-layered ring on periodic radial supports.

$$-K_1\left[\left(u+\frac{\partial v_1}{\partial\theta}\right)\delta(v_1)\right]\Big|_0^\lambda=0 \tag{6b}$$

$$-K_3\left[\left(u+\frac{\partial v_3}{\partial\theta}\right)\delta(v_3)\right]\Big|_0^\lambda$$

$$+\frac{1}{2}\frac{k_r}{R_3^2}\left\{\left[\left(\frac{\partial u}{\partial\theta}-v_3\right)\delta(v_3)\right]\Big|_{\theta=0}\right.$$

$$\left.+\left[\left(\frac{\partial u}{\partial\theta}-v_3\right)\delta(v_3)\right]\Big|_{\theta=\lambda}\right\}=0 \tag{6c}$$

$$\left\{\left[D\left(\frac{\partial u}{\partial\theta}+\frac{\partial^3u}{\partial\theta^3}\right)-gd_2\left(v_1d_1-v_3d_3+\frac{\partial u}{\partial\theta}d_2\right)\right.\right.$$

$$\left.\left.-\frac{m_2}{2}d_4\left(\frac{\partial^2v_1}{\partial t^2}d_1+\frac{\partial^2v_3}{\partial t^2}d_3+\frac{\partial^3u}{\partial\theta\partial t^2}d_4\right)\right]\delta(u)\right\}\Big|_0^\lambda=0 \tag{6d}$$

where $m=m_1+m_2+m_3$ and $g=Gb/2\delta_2$.

The differential equations (5) are different from the ones given in Ref. 6, as the tangential inertia of the core is also included in the present work.

From the boundary conditions [Eqs. (6)], the harmonic amplitudes of the generalized coupling forces (at either end of a bay) of the periodic element corresponding to the coupling coordinates, q_1 $(=\partial u/\partial\theta)$, $q_2(=v_1)$, $q_3(=v_3)$, and $q_4(=u)$ can be identified as

$$\bar{Q}_{10}=\left[-D\left(U+\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}\right)+\frac{k_r}{2R_3^2}\left(\frac{\mathrm{d}U}{\mathrm{d}\theta}-V_3\right)\right]\Big|_{\theta=0} \tag{7a}$$

$$\bar{Q}_{1\lambda}=\left[-D\left(U+\frac{\mathrm{d}^2U}{\mathrm{d}\theta^2}\right)-\frac{k_r}{2R_3^2}\left(\frac{\mathrm{d}U}{\mathrm{d}\theta}-V_3\right)\right]\Big|_{\theta=\lambda} \tag{7b}$$

$$\bar{Q}_{20}=\left[-K_1\left(U+\frac{\mathrm{d}V_1}{\mathrm{d}\theta}\right)\right]\Big|_{\theta=0} \tag{7c}$$

$$\bar{Q}_{2\lambda} = \left[-K_1 \left(U + \frac{dV_1}{d\theta} \right) \right] \Big|_{\theta=\lambda} \tag{7d}$$

$$\bar{Q}_{30} = \left[-K_3 \left(U + \frac{dV_3}{d\theta} \right) - \frac{k_r}{2R_3^2} \left(\frac{dU}{d\theta} - V_3 \right) \right] \Big|_{\theta=0} \tag{7e}$$

$$\bar{Q}_{3\lambda} = \left[-K_3 \left(U + \frac{dV_3}{d\theta} \right) + \frac{k_r}{2R_3^2} \left(\frac{dU}{d\theta} - V_3 \right) \right] \Big|_{\theta=\lambda} \tag{7f}$$

$$\bar{Q}_{40} = \left[D \left(\frac{dU}{d\theta} + \frac{d^3U}{d\theta^3} \right) - gd_2 \left(V_1 d_1 - V_3 d_3 + \frac{dU}{d\theta} d_2 \right) + \frac{m_2}{2} \omega^2 d_4 \left(V_1 d_1 + V_3 d_3 + \frac{dU}{d\theta} d_4 \right) \right] \Big|_{\theta=0} \tag{7g}$$

$$\bar{Q}_{4\lambda} = \left[D \left(\frac{dU}{d\theta} + \frac{d^3U}{d\theta^3} \right) - gd_2 \left(V_1 d_1 - V_3 d_3 + \frac{dU}{d\theta} d_2 \right) + \frac{m_2}{2} \omega^2 d_4 \left(V_1 d_1 + V_3 d_3 + \frac{dU}{d\theta} d_4 \right) \right] \Big|_{\theta=\lambda} \tag{7h}$$

where

$$Q_{j0} = \bar{Q}_{j0} e^{i\omega t}, \quad Q_{j\lambda} = \bar{Q}_{j\lambda} e^{i\omega t}, \quad (j=1,2,3,4)$$

$$u = U e^{i\omega t}, \quad v_1 = V_1 e^{i\omega t}, \text{ and } v_3 = V_3 e^{i\omega t}$$

with ω as the frequency.

The direction of Q_{j0} is assumed to be opposite to that of $Q_{j\lambda}$ with the subscripts 0 and λ referring to the values at $\theta=0$ and λ, respectively.

It may be noted that the maximum number of coupling coordinates for the three-layered ring is four.

B. Natural Frequencies of an Elastic Ring

Theory of Wave Propagation and Natural Frequencies

The free harmonic vibration of an infinite, periodic, undamped structure can be analyzed into a special type of wave motion.[15] These free waves can propagate through the structure without decaying only in certain frequency zones called "propagation bands." Outside these bands, the waves decay as they spread outward. This motion is characterized by a complex constant μ called the propagation constant. The real (μ_r) and imaginary (μ_i) parts of this constant are the measures of the rate of decay and the change of phase, respectively, over the distance between the adjacent supports. For the structure under consideration, this implies that

$$e^{\mu} q_j \big|_{\theta=0} = q_j \big|_{\theta=\lambda} \quad \text{and} \quad e^{\mu} Q_j \big|_{\theta=0} = Q_j \big|_{\theta=\lambda} \quad (j=1,2,\ldots) \tag{8}$$

The propagation constants always occur in pairs (positive and negative) and the number of pairs is equal to the minimum number of coupling coordinates between the adjacent periodic elements.[16]

For the ring under consideration, the special type of harmonic waves propagate without any reflection as the structure does not have any ends. Hence, the ring can be considered as an infinite periodic structure except that the propagating waves should have the same phase and magnitude after traveling once around the complete ring. Consequently, the natural frequencies ω_n at which harmonic waves can propagate are not given by the entire propagation band but are limited to a discrete set[17] given by

$$\mu_{rn} = 0 \text{ and } \mu_{in} = \pm 2j\pi/N \quad (j=0,1,\ldots N) \tag{9}$$

where N is the number of bays in the ring and the subscript n refers to the value at a natural frequency.

Different Support Conditions

For the first type of support conditions, the number of coupling coordinates is two (i.e., $q_1 = \partial u/\partial\theta$ and $q_2 = v_1$) and the system is bicoupled. The amplitudes of the associated harmonic coupling forces are given by Eqs. (7a-7d). The propagation constant curves for this system are obtained by the receptance method.[12,17] From these propagation constant curves, the natural frequencies, Ω_n $(=\omega_n\sqrt{m_3/D_3})$, can be obtained by using Eq. (9).

When the supports are considered to prevent only the radial displacement, the ring becomes a tricoupled system with coupling coordinates $q_1 = \partial u/\partial\theta$, $q_2 = v_1$, and $q_3 = v_3$. For this system, the amplitudes of the associated harmonic coupling forces are given by Eqs. (7a-7f). Then the natural frequencies can be obtained by the same procedure as used for the bicoupled system.

C. Resonance Frequencies and Loss Factors of a Damped Ring

In contrast to the previous section, here the core (middle layer) is considered to be viscoelastic with a loss factor β, i.e., its shear modulus becomes complex and is of the form $G^* = G(1+i\beta)$. Then the ring becomes a damped structure and the free waves can no longer propagate without decay at any frequency. To find the resonance frequencies Ω_n and the loss factors η of the composite structure, the theory of "forced damped normal modes"[13] is used. These forced damped normal modes are obtained by exciting the structure with a loading that is harmonic and in phase with the local velocity but proportional in amplitude to the local inertia. This is achieved by replacing ω^2 by $\omega^{*2} = \omega^2(1+i\eta)$ in Eqs. (5a-5c).

For the harmonic variation of the displacements u, v_1, and v_3, Eqs. (5a-5c) become, respectively,

$$L_1 U + L_2 V_1 + L_3 V_3 = 0 \tag{10a}$$

$$L_2 U + L_4 V_1 + L_5 V_3 = 0 \tag{10b}$$

$$L_3 U + L_5 V_1 + L_6 V_3 = 0 \tag{10c}$$

In the above equations,

$$L_1 = I_1 \frac{d^4}{d\theta^4} + I_2 \frac{d^2}{d\theta^2} + I_3, \quad L_2 = I_4 \frac{d}{d\theta}$$

$$L_3 = I_5 \frac{d}{d\theta}, \quad L_4 = I_6 \frac{d^2}{d\theta^2} + I_7$$

$$L_5 = I_8, \quad L_6 = I_9 \frac{d^2}{d\theta^2} + I_{10}$$

where I_j are defined by Eq. (A1) in the Appendix.

To compute the resonance frequencies and the associated modal loss factors, the general solution of Eqs. (10) can be written in the form

$$\bar{U}(\theta) = \sum_{j=1}^{8} C_j e^{s_j\theta}, \quad \bar{V}_1(\theta) = \sum_{j=1}^{8} B_j e^{s_j\theta}, \quad \bar{V}_3(\theta) = \sum_{j=1}^{8} \bar{B}_j e^{s_j\theta} \tag{11}$$

with $\bar{U} = U/R_3$, $\bar{V}_1 = V_1/R_3$, and $\bar{V}_3 = V_3/R_3$.

The s_j are the eight roots of the auxiliary equation

$$s^8 + a_1 s^6 + a_2 s^4 + a_3 s^2 + a_4 = 0 \tag{12}$$

where a_1, a_2, a_3, and a_4 are defined by Eq. (A2).

For the first type of support (the bicoupled system), the following conditions must be satisfied:

1) The radial displacement u is zero at $\theta=0$ and λ, i.e.,

$$\sum_{j=1}^{8} C_j=0 \tag{13a}$$

and

$$\sum_{j=1}^{8} C_j e^{s_j\lambda}=0 \tag{13b}$$

2) The tangential displacement of the base layer (layer 3) v_3 is zero at $\theta=0$ and λ, i.e.,

$$\sum_{j=1}^{8} C_j \bar{T}_j=0 \tag{13c}$$

and

$$\sum_{j=1}^{8} C_j \bar{T}_j e^{s_j\lambda}=0 \tag{13d}$$

where $\bar{T}_j=\bar{B}_j/C_j$ and the expression for it is given by Eq. (A3).

3) The coupling coordinate $\partial u/\partial\theta$ and the associated coupling force Q_1 must satisfy the wave condition, i.e.,

$$e^{i\mu_{in}}\frac{\partial u}{\partial\theta}\bigg|_{\theta=0}=\frac{\partial u}{\partial\theta}\bigg|_{\theta=\lambda}$$

or

$$\sum_{j=1}^{8} C_j s_j\,(e^{i\mu_{in}}-e^{s_j\lambda})=0 \tag{13e}$$

and

$$e^{i\mu_{in}}Q_{10}=Q_{1\lambda}$$

or

$$\sum_{j=1}^{8} C_j\left[-\frac{D}{D_3}s_j^2(e^{i\mu_{in}}-e^{s_j\lambda})+\bar{k}_r s_j e^{s_j\lambda}\right]=0 \tag{13f}$$

where $\bar{k}_r$ $(=k_r/D_3R_3^2)$ is the nondimensional rotational stiffness.

4) The coupling coordinate v_1 and the associated coupling force Q_2 must satisfy the wave condition, i.e.,

$$e^{i\mu_{in}}v_1\,|_{\theta=0}=v_1\,|_{\theta=\lambda}$$

or

$$\sum_{j=1}^{8} C_j T_j\,(e^{i\mu_{in}}-e^{s_j\lambda})=0 \tag{13g}$$

and

$$e^{i\mu_{in}}Q_{20}=Q_{2\lambda}$$

or

$$\sum_{j=1}^{8} C_j T_j s_j\,(e^{i\mu_{in}}-e^{s_j\lambda})=0 \tag{13h}$$

where $T_j=B_j/C_j$ and the expression for it is given by Eq. (A4).

Equations (13) are eight homogeneous equations in eight unknowns (C_j). For the nontrivial solution of these unknowns, the coefficient determinant should vanish. Using Eq. (12) along with this condition, the resonance frequencies Ω_n and the associated modal loss factors η of the composite structure can be determined by a two-dimensional Newton-Raphson method.[14] Quick and satisfactory convergence of this iterative procedure was found to be insured by choosing the initial values of Ω_n as somewhat higher ($\approx 10\%$) than the natural frequencies obtained from the corresponding elastic analysis (i.e., $\beta=0$) outlined in Sec. II.B.

When the ring is considered to be tricoupled, the conditions to be satisfied are:

1) The radial displacement u is zero at $\theta=0$ and λ, i.e.,

$$\sum_{j=1}^{8} C_j=0 \tag{14a}$$

and

$$\sum_{j=1}^{8} C_j e^{s_j\lambda}=0 \tag{14b}$$

2) The coupling coordinate $\partial u/\partial\theta$ and the coupling force Q_1 must satisfy the wave condition, i.e.,

$$\sum_{j=1}^{8} C_j s_j\,(e^{i\mu_{in}}-e^{s_j\lambda})=0 \tag{14c}$$

and

$$\sum_{j=1}^{8} C_j\left[-\frac{D}{D_3}s_j^2(e^{i\mu_{in}}-e^{s_j\lambda})+\bar{k}_r(s_j-\bar{T}_j)e^{s_j\lambda}\right]=0 \tag{14d}$$

3) The coupling coordinate v_1 and the coupling force Q_2 must satisfy the wave condition, i.e.,

$$\sum_{j=1}^{8} C_j T_j\,(e^{i\mu_{in}}-e^{s_j\lambda})=0 \tag{14e}$$

and

$$\sum_{j=1}^{8} C_j T_j s_j\,(e^{i\mu_{in}}-e^{s_j\lambda})=0 \tag{14f}$$

4) The coupling coordinate v_3 and the coupling force Q_3 must satisfy the wave condition, i.e.,

$$e^{i\mu_{in}}v_3\,|_{\theta=0}=v_3\,|_{\theta=\lambda}$$

or

$$\sum_{j=1}^{8} C_j \bar{T}_j\,(e^{i\mu_{in}}-e^{s_j\lambda})=0 \tag{14g}$$

and

$$e^{i\mu_{in}}Q_{30}=Q_{3\lambda}$$

or

$$\sum_{j=1}^{8} C_j\left[-\frac{K_3}{D_3}s_j\bar{T}_j(e^{i\mu_{in}}-e^{s_j\lambda})-\bar{k}_r(s_j-\bar{T}_j)e^{s_j\lambda}\right]=0 \tag{14h}$$

Again, Eqs. (14) constitute a set of eight homogeneous equations for the tricoupled system and hence the resonance frequencies and the loss factors can be computed for this type of support as well.

D. Natural Modes of an Elastic Ring

With the numerical data considered in the present work, the number of pairs of propagating waves that give rise to natural modes is found to be only one. Based on this fact, natural modes of the composite elastic ring are computed. Similar to the case of a two-layered ring,[12] the natural frequencies for which $\mu_{in}\neq 0$ and $\mu_{in}=0$ are treated separately, as the former leads to degenerate modes whereas the latter gives rise to unique modes.

For $\mu_{in}\neq 0$, the modes being degenerate, two modes (antisymmetric and symmetric with respect to a diameter passing through a support) are obtained by prescribing a coupling force at the supports. With $\mu_{in}=0$, the problem is formulated as an eigenvalue problem.[12]

III. Results and Discussion

Numerical results are presented for the following data: $\rho_1/\rho_3=1.0$, $\rho_2/\rho_3=0.2$, $E_1/E_3=1.0$, $G/E_3=0.0001$, $N=3$ (i.e., $\lambda=2\pi/3$), and $\beta=1.0$. The effects of the rotational constraint $\bar{k}_r$ at the supports were found to be similar to those observed in the case of a two-layered ring.[11] So the results are presented here only for $\bar{k}_r=0$.

A. Elastic Core

Figures 2 and 3 show the propagation constant curves for the bicoupled and tricoupled systems, respectively, with $\delta_1=0.015$, $\delta_2=0.03$, and $\delta_3=0.03$. The nature of the curves in Fig. 2 is somewhat different from that of a two-layered ring (monocoupled system).[11] This difference arises because of the additional propagation constant. This propagation constant may be associated with the extra degree of freedom, i.e., the tangential displacement of the outer layer (layer 1) v_1. By reducing the thickness of the core δ_2 and consequently curtailing the degree of freedom v_1, it was found that this difference appears only at higher frequencies.

The data presented above indicate a rather thick core and a low value of the shear modulus parameter (G/E_3). Under such conditions, a predominant tangential motion of the outer layer is expected. Thus, the natural frequency obtained ($\Omega_n=11.768$, Ω_{n3} in Fig. 2) due to the additional propagation constant is expected to be associated with the tangential motion. A confirmation to this effect can be found from the mode shapes or from a comparison of the various energies at this frequency, which will be discussed later. However, the other natural frequencies (1, 2, and 4) do not exhibit any such characteristic. Also in Fig. 3, a similar argument applies and the mode at the fifth natural frequency ($\Omega_n=15.550$) can be said to be associated with predominant tangential motion of the outer layer. Hereafter, these modes are referred to as "tangential" and the others as "flexural."

At the first few natural frequencies (flexural), the mode shapes were found to be very similar to those of single- and two-layered rings[12,17] and hence are not shown here. If the radial displacement pattern in a mode is symmetric about a diameter passing through a support, then the corresponding tangential displacement pattern is antisymmetric about the same diameter and vice versa. The various strain energies of different layers in these modes were also computed. The ratios of these to the total bending strain energy are presented in Table 1.

Comparing the relative magnitudes of the displacements, it is observed that the ratios of (radial/tangential) displacements in the first few flexural modes are of the order of 1.1-1.5. This is true for both bicoupled and tricoupled systems. On the other hand, these ratios in the tangential modes are approximately 0.3 for the bicoupled system and exactly zero for the tricoupled system. Further discussion on these tangential modes is presented separately for the two systems.

For the bicoupled system, the tangential mode is associated with small radial displacements. Moreover, this radial

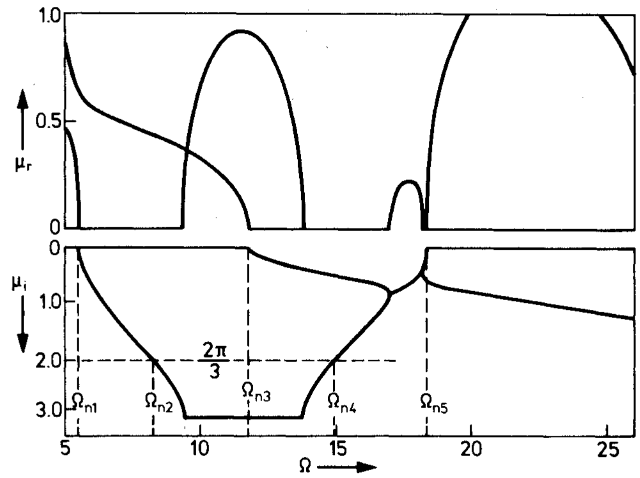

Fig. 2 Variation of propagation constant with frequency, bicoupled system.

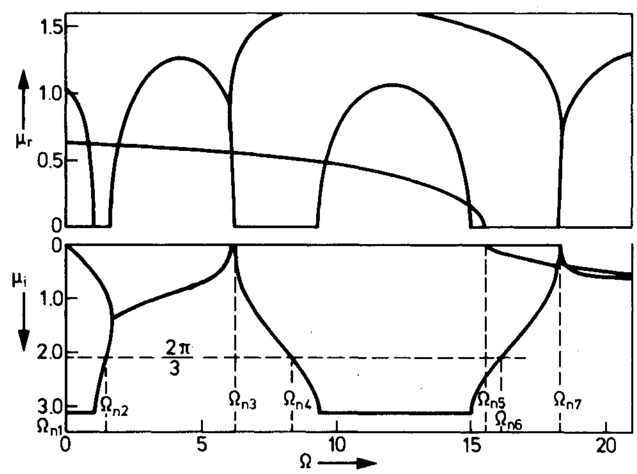

Fig. 3 Variation of propagation constant with frequency, tricoupled system.

Table 1 Natural frequencies and various energies of an elastic three-layered ring ($\delta_1=0.015$, $\delta_2=\delta_3=0.03$, $\bar{k}_r=0$, $G/E_3=0.0001$, $\rho_1/\rho_3=1$, $\rho_2/\rho_3=0.2$, $N=3$)

Mode No.	Phase constant, μ_{in}	Natural frequency, Ω_n	Strain energy/total bending strain energy: Extensional, Layer 1	Extensional, Layer 3	Shear, Layer 2
			Bicoupled system		
1	0	5.522	3.36×10^{-4}	4.74×10^{-3}	1.57×10^{-1}
2	$2\pi/3$	8.262	1.92×10^{-3}	2.57×10^{-3}	6.09×10^{-2}
3	0	11.768	1.07×10^{-3}	2.09×10^{-1}	2.26×10^{-1}
4	$2\pi/3$	14.920	7.02×10^{-4}	3.91×10^{-2}	4.26×10^{-2}
			Tricoupled system		
2	$2\pi/3$	1.452	4.29×10^{-3}	4.50×10^{-3}	3.29×10^{-1}
3	0	6.230	2.68×10^{-4}	9.14×10^{-4}	1.14×10^{-1}
4	$2\pi/3$	8.325	1.42×10^{-3}	2.84×10^{-3}	6.19×10^{-2}
5	0	15.550	∞	∞	∞

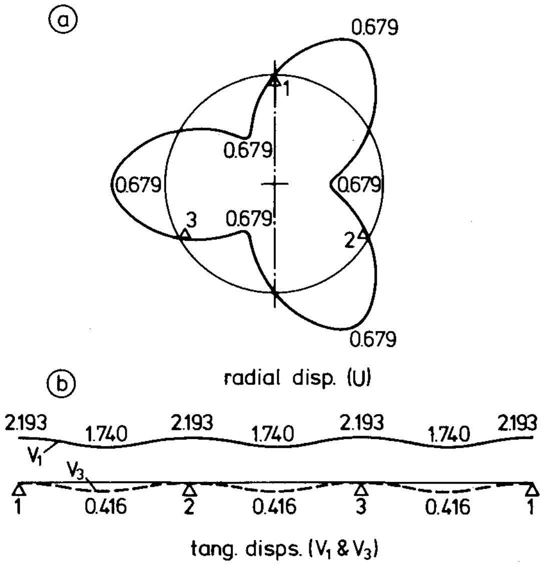

Fig. 4 Mode shapes of three-layered ring, bicoupled system.

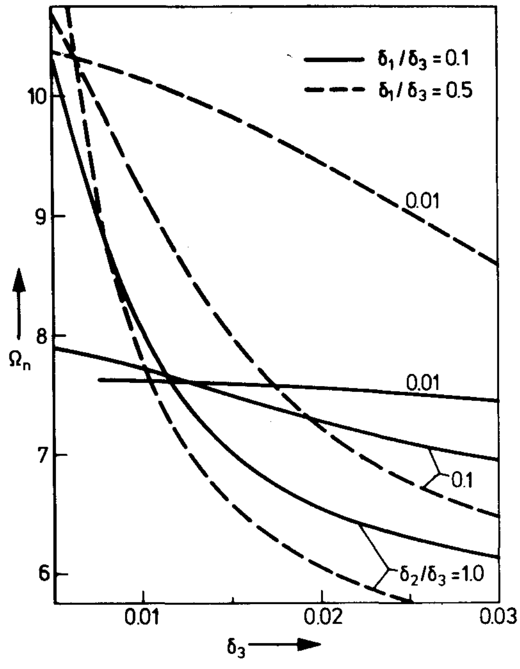

Fig. 5 Variation of resonance frequency with base layer thickness (mode 1), bicoupled system.

displacement pattern (Fig. 4a) is very similar to that of the nearest flexural mode having $\mu_{in}=0$ as the tangential mode also corresponds to zero phase difference. From Fig. 4b it can be seen that the difference in the tangential displacements $(V_1 - V_3)$ of this mode is nearly independent of θ.

For the tricoupled system, however, the tangential mode is not associated with any radial displacement and both the displacements V_1 and V_3 remained independent of θ $(V_1/V_3 = -1.418)$. It may be noted that the natural frequency and the relative magnitudes of V_1 and V_3 (i.e., V_1/V_3) of this mode can be verified by substituting the solution $U=0$, $V_1=B$, and $V_3=\bar{B}$ (B and $\bar{B}$ are constants) in the differential equations (10). Hence, this mode is nothing but a torsional oscillation of the ring about its center in which layers 1 and 3 move in counterphase without any radial deformation anywhere. In other words, at this mode the system behaves like a torsional system consisting of two disks connected by a shaft.

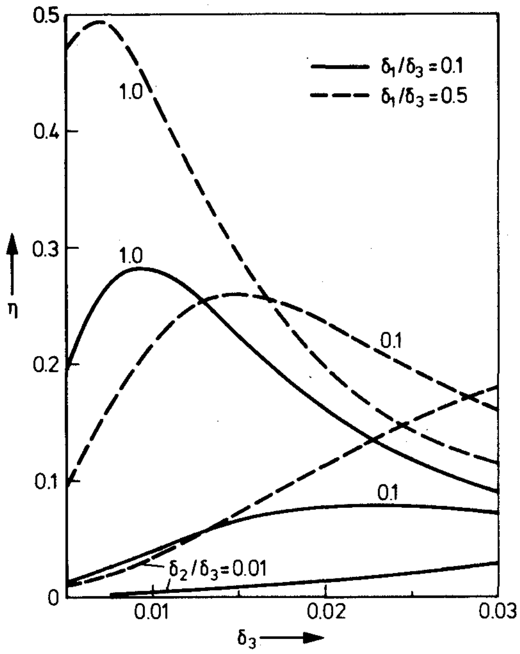

Fig. 6 Variation of modal loss factor with base layer thickness (mode 1), bicoupled system.

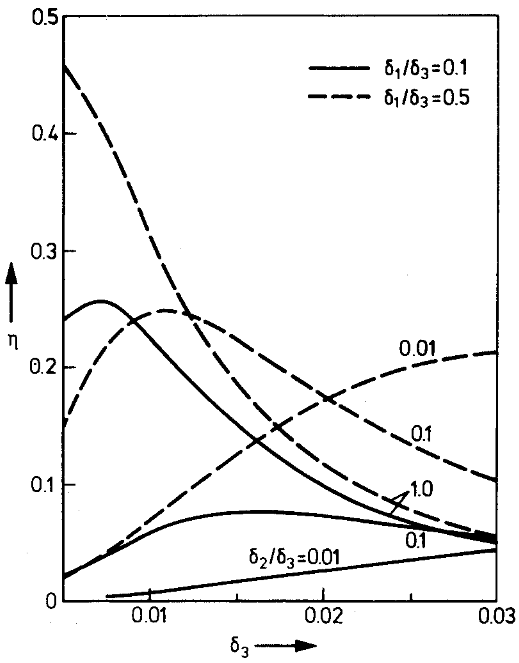

Fig. 7 Variation of modal loss factor with base layer thickness (mode 2), bicoupled system.

For both systems, v_1 and v_3 are out of phase, giving rise to a large shear deformation in the core. As mentioned earlier, it can be seen from Table 1 that the shear strain energies in these modes are large compared to the bending and extensional strain energies. Moreover, the table indicates that the third and fourth natural frequencies in the tricoupled system are comparable to the first and second ones, respectively, in the bicoupled system. The deformation pattern and the values of strain energies at the corresponding frequencies are also seen to be similar.

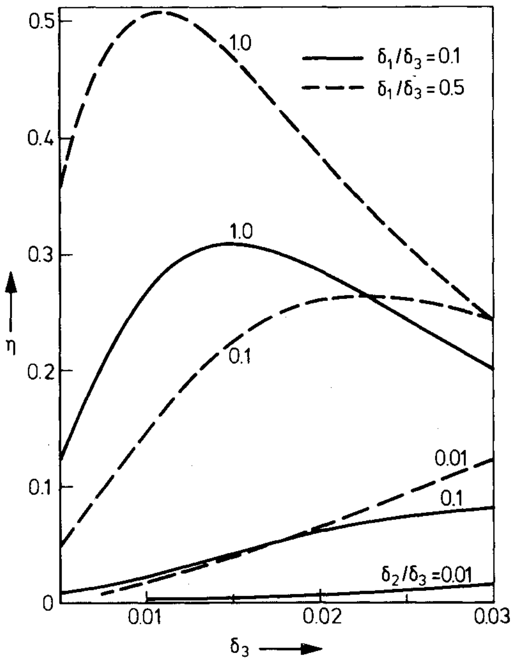

Fig. 8 Variation of modal loss factor with base layer thickness (mode 2), tricoupled system.

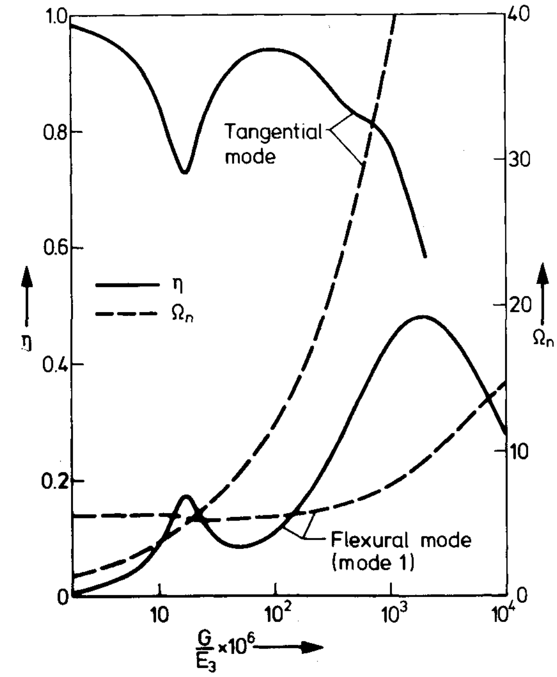

Fig. 9 Variation of modal loss factor and resonance frequency with core shear modulus, bicoupled system.

Table 2 Resonance frequencies and loss factors of a damped three-layered ring ($\delta_1 = 0.015$, $\delta_2 = \delta_3 = 0.03$, $k_r = 0$, $G/E_3 = 0.0001$, $\rho_1/\rho_3 = 1$, $N = 3$, $\rho_2/\rho_3 = 0.2$, $\beta = 1.0$)

Mode No.	Phase constant, μ_{in}	Resonance frequency, Ω_n	Loss factor, η
	Bicoupled system		
1	0	5.567	0.1143
2	$2\pi/3$	8.265	0.0571
3	0	11.850	0.9417
4	$2\pi/3$	14.921	0.0284
	Tricoupled system		
2	$2\pi/3$	1.458	0.2436
3	0	6.236	0.1019
4	$2\pi/3$	8.327	0.0580
5	0	15.552	1.0000

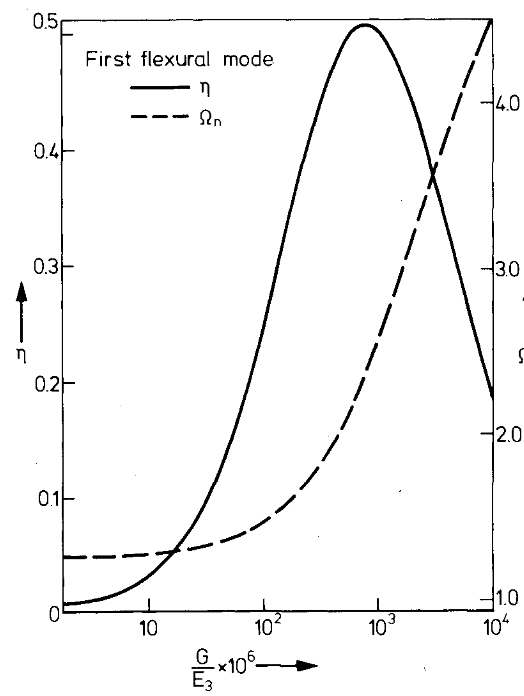

Fig. 10 Variation of modal loss factor and resonance frequency with core shear modulus, tricoupled system.

B. Viscoelastic Core

Results are presented with the assumption that the core loss factor β is frequency independent. However, variations of β with the frequency can be easily incorporated into the computational procedure.

Figure 5 shows the variation of the resonance frequency Ω_n with the base layer thickness δ_3 for the bicoupled system (mode 1). At the other modes, the trends were seen to be very similar. At low values of δ_1/δ_3 and δ_2/δ_3, the rate of decrease of Ω_n with δ_3 is very small, whereas with higher values of those parameters, resonance frequencies decrease steeply as δ_3 increases. The reason for this behavior is that, for the data considered, the effective stiffness of the structure does not change at the same rate as the inertia.

The resonance frequencies were seen to increase with increasing value of β. However, this effect is marginal and depends on the thicknesses of the core and the constraining layer. The presence of rotational stiffness at the supports k_r was also found to increase Ω_n as expected.

Figures 6-8 show the variation of the modal loss factor η in the first two flexural modes. There exists an optimum value of δ_3 at which η is maximum. These optimum values are referred to as δ_3^* and η^*, respectively. The values of δ_3^* and η^* are found to be very sensitive to both δ_1/δ_3 and δ_2/δ_3. For low values of the core thickness, the optimum value δ_3^* turns out to be very large. As expected, with a thicker core, i.e., with a decreasing value of δ_3^*, the value of η^* increases. However, for a given value of δ_3, the core and the constraining layer thicknesses have to be chosen so as to yield a maximum possible loss factor. This is obvious, because for a given value of δ_1/δ_3, the loss factor curves for various values of δ_2/δ_3 intersect each other at different points. Furthermore, the curves for different combinations of δ_2/δ_3 and δ_1/δ_3 are also seen to intersect. All of these trends imply that both the core and the constraining layer thicknesses govern the modal loss factors of the composite ring in a complicated manner. Comparing Figs. 6 and 7, it is apparent that at higher modes the optimum value δ_3^* shifts to lower values.

For the tricoupled system, the nature of the variation in the loss factors in the third mode has not been shown. This was found to be similar to that in the first mode of the bicoupled system. This is so because these modes are comparable and is explained in connection with the elastic ring. On the other hand, the loss factors in the second mode, shown in Fig. 8, have wider peaks. This gives the designer the flexibility to choose different thicknesses of the layers, without sacrificing the loss factor in the second mode.

The resonance frequencies and the modal loss factors are given in Table 2. It can be seen that the third and the fourth resonance frequencies of the tricoupled system are comparable to the first and the second ones, respectively, of the bicoupled system. In these modes the loss factors are also nearly the same. For the bicoupled system, the loss factor η in the tangential mode (No. 3) is almost equal to the core loss factor β. This is expected, because from Table 1 it is seen that the shear strain energy in this mode is quite high. Likewise, for the tricoupled system, η in the tangential mode (No. 5) is exactly equal to β. This is because the supports in this case do not prevent the tangential movement of the base layer and thus allow the core to deform entirely in shear without any other deformation anywhere.

Figures 9 and 10 show the effect of the core shear modulus G/E_3 on the loss factor η and on the resonance frequency Ω_n in the first flexural mode. The values of all other parameters for these curves are given in Table 2. For the tricoupled system, the curves (Fig. 10) are similar to the ones reported for a damped three-layered unsupported ring.[7] Here, Ω_n increases with increase in G/E_3, whereas the loss factor reaches a maximum for an optimum value of G/E_3. This is because at higher values of G/E_3 the shear deformation is less, and at lower values of G/E_3 the damping force is less. In both the cases, the resulting dissipation of energy is reduced and maximum energy is dissipated at an intermediate value of G/E_3.

The loss factor curve for the bicoupled system shows two peaks (Fig. 9). The second peak, having a large value of η, occurs at the optimum value of G/E_3 discussed above. In order to explain the presence of the first peak, the resonance frequency and loss factor of the tangential mode are also shown in Fig. 9. At the value of G/E_3 where the first peak in the flexural mode occurs, it can be seen that the resonance frequencies of flexural and tangential modes coincide, resulting in a coupling of these modes. Due to the coupling, modes near this frequency are neither predominantly tangential nor predominantly flexural. As a result, the loss factor in the flexural mode has a peak, whereas that in the tangential mode shows a dip. This type of coupling is not seen in the tricoupled system because the flexural mode considered in Fig. 10 corresponds to $\mu_{in} = 2\pi/3$ (mode 2), whereas the first tangential mode corresponds to a different value of $\mu_{in} (=0)$.

IV. Conclusions

The theory of wave propagation can be used efficiently for the vibration analysis of damped sandwiched rings on periodic radial supports. The computational effort is independent of the number of supports. The results for the corresponding elastic ring have been found to be useful for both computation and interpretation of the results of the damped ring. The composite loss factors are high in the modes associated with predominant tangential displacements. For a given shear modulus of the core, there exists an optimum combination of the thicknesses of the various layers, which yields the maximum modal loss factor. There is also an optimum value of the core shear modulus (other parameters remaining the same) that gives rise to a maximum modal loss factor. With a certain combination of the parameters, there may be coupling between "tangential" and "flexural" modes. In these situations, the loss factor in the "flexural" mode shows a peak and that in the "tangential" mode exhibits a minimum.

Appendix

$$I_1 = D/D_3$$

$$I_2 = \frac{2D}{D_3} - \frac{g^* d_2^2}{D_3} + \frac{\Omega^{*2}}{2}\frac{m_2}{m_3} d_4^2$$

$$I_3 = \frac{D}{D_3} + \frac{K_1}{D_3} + \frac{K_3}{D_3} - \Omega^{*2}\frac{m}{m_3}$$

$$I_4 = \frac{K_1}{D_3} - \frac{g^* d_1 d_2}{D_3} + \frac{\Omega^{*2}}{2}\frac{m_2}{m_3} d_1 d_4$$

$$I_5 = \frac{K_3}{D_3} + \frac{g^*}{D_3} d_2 d_3 + \frac{\Omega^{*2}}{2}\frac{m_2}{m_3} d_3 d_4$$

$$I_6 = K_1/D_3$$

$$I_7 = \Omega^{*2}\left(\frac{m_1}{m_3} + \frac{1}{2}\frac{m_2}{m_3} d_1^2\right) - \frac{g^*}{D_3} d_1^2$$

$$I_8 = \left(\frac{g^*}{D_3} + \frac{\Omega^{*2}}{2}\frac{m_2}{m_3}\right) d_1 d_3$$

$$I_9 = K_3/D_3$$

$$I_{10} = \Omega^{*2}\left(1 + \frac{1}{2}\frac{m_2}{m_3} d_3^2\right) - \frac{g^*}{D_3} d_3^2 \tag{A1}$$

where $g^* = g(1 + i\beta)$.

The coefficients of the auxiliary equation (12) are given by

$$a_1 = (I_1 I_6 I_{10} + I_1 I_7 I_9 + I_2 I_6 I_9)/I_1 I_6 I_9$$

$$a_2 = (I_1 I_7 I_{10} + I_3 I_6 I_9 + I_2 I_6 I_{10} + I_2 I_7 I_9 - I_1 I_8^2 - I_4^2 I_9 - I_5^2 I_6)/I_1 I_6 I_9$$

$$a_3 = (I_2 I_7 I_{10} + I_3 I_6 I_{10} + I_3 I_7 I_9 - I_2 I_8^2 - I_4^2 I_{10} - I_5^2 I_7 + 2 I_4 I_5 I_8)/I_1 I_6 I_9$$

$$a_4 = (I_3 I_7 I_{10} - I_3 I_8^2)/I_1 I_6 I_9 \tag{A2}$$

where the I_j are given by Eq. (A1).

The expressions for $\bar{T}_j$ and T_j can be written as

$$\bar{T}_j = [-I_5 I_6 s_j^3 + (I_4 I_8 - I_5 I_7) s_j]/a \tag{A3}$$

and

$$T_j = [-I_4 I_9 s_j^3 + (I_5 I_8 - I_4 I_{10}) s_j]/a \tag{A4}$$

where $a = I_6 I_9 s_j^4 + (I_6 I_{10} + I_7 I_9) s_j^2 + (I_7 I_{10} - I_8^2)$, and the I_j are again given by Eq. (A1).

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